Noncommutative Chern-Simons solitons

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The Chern-Simons theories on a noncommutative plane, which are shown to describe the quantum Hall liquid, are considered. We introduce matter fields fundamentally coupled to the noncommutative Chern-Simons field. Exploiting BPS equations for the nonrelativistic Chern-Simons theory, we find the exact solutions of multivortices that are closely packed and exponentially localized. We determine the position, the size, and the angular momentum explicitly. We then construct solutions of two spatially separated vortices, and determine the moduli dependence of the size and the angular momentum. We also consider the relativistic Chern-Simons theory and find nontopological solutions whose properties are similar to the nonrelativistic counterpart. However, unlike the nonrelativistic case, there are two branches of solutions for a given magnetic field and they cease to exist below a certain noncommutativity scale.

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I. INTRODUCTION

The solitons arising in noncommutative theories have attracted much attention recently [1–17]. These include the noncommutative monopoles [2,3,11], the vortices [5,8,14], the unstable lower-dimensional D-branes [9,10,16,17], and so on. As in the case of $U(2)$ noncommutative monopoles [2,11], some of the solutions are smoothly connected to the commutative ones. There are noncommutative solitons such as $(2+1)$-dimensional scalar noncommutative solitons [1], which cease to exist in the commutative limit. In the latter case, the characteristic properties of the noncommutative solitons largely differ from the ordinary solitons and have not been fully explored yet.

Various aspects of the ordinary Chern-Simons solitons are well known thus far [18–20]. In this paper, we would like to explore the solitons in the noncommutative Chern-Simons theories. (Related discussions of noncommutative Chern-Simons theories have appeared in Refs. [21–28].) It was shown recently that the quantum Hall liquid can be described by Chern-Simons theories on the noncommutative plane [29]. We shall add fundamental matter fields to the system, which may be viewed as describing the density fluctuations of the quantum Hall liquid.

We first consider the nonrelativistic Chern-Simons theory on the noncommutative plane, which reduces to the Jackiw-Pi model in the commutative limit [19]. We solve the Bogomel'nyi-Prasad-Sommerfield (BPS) equations and find the exact solutions of multi-vortices, which are localized exponentially, carrying integer flux $\Phi=m>0$. This is quite contrasted to the case of the commutative Jackiw-Pi solitons where the matter density falls off only in a certain power of the radius. To study the properties of the noncommutative solitons, we employ the covariant position operator [30,31] that describes the real space location of the fluid particles [29]. With help of this operator, we are able to define gauge invariant position, size and angular momentum of the vortices. Hence it is extremely useful to explore the ''local'' properties of generic profiles in spite of the fact that all the gauge invariant quantities in the noncommutative gauge theories more or less require integration over space.[1] In particular, the moduli size of these noncommutative Chern-Simons vortices is found to be $\sqrt{\theta(m-1)}$ and the total angular momentum $\pi\kappa m(m-2)$ where θ is the noncommutativity parameter. The intervortex distance is roughly order of $\sqrt{\theta}$ and cannot be squeezed further beyond the distance. This can be understood as revealing the area preserving nature of the quantum Hall liquid. Thus the simple exact solutions describe the closely packed m vortices.

We construct explicitly the two vortex solutions separated spatially. There are four real moduli parameters, which describe the positions of each vortex. We further identify the moduli dependence of the size of the two vortex configuration as well as the angular momentum. Identifying the angular momentum at infinite separation as an intrinsic one, the

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[1]The Seiberg-Witten map is a way to find the local gauge invariant quantities [32].

orbital part of angular momentum is found to be $\pi\kappa m(m-1)$ in the case of the closely packed vortices.

We then study the general solutions of the BPS equations within the rotationally symmetric ansatz. We show that the $\Phi=1$ solution above is unique within the ansatz. Except this $\Phi=1$ solution, we have proven that all the other solutions within the ansatz ought to carry the flux whose absolute value is greater than unity. Hence it seems clear that the minimal flux in the theory is one for the solitonic sector.

Next we consider the relativistic Chern-Simons model [18] on the noncommutative plane. We construct the nontopological solutions, which are quite similar to the nonrelativistic noncommutative vortices. However, unlike the nonrelativistic model, there are two branches of the nontopological solutions for the same magnetic field and these solutions cease to exist when the noncommutativity parameter becomes small. We also identify the size and the angular momentum and discuss the separation of two vortices.

In Sec. II, we describe the nonrelativistic Chern-Simons model and derive the BPS equations. In Sec. III, we present the closely packed multi-vortex solutions and discuss their properties. The separations of two closely packed vortices are studied in Sec. IV by constructing explicitly the solutions depending on the moduli parameters. In Sec. V, we prove that, within the rotationally symmetric ansatz, the flux is limited to $\Phi\geqslant 1$ or $\Phi<-1$. In Sec. VI, we derive the BPS equation for the relativistic Chern-Simons model and construct the nontopological vortex solutions. The last section comprises concluding remarks and discussions.

II. NONCOMMUTATIVE CHERN-SIMONS THEORIES

The noncommutative Chern-Simons theories are described by the Lagrangian

$$L_{\rm CS}=-\frac{\kappa}{2}\int d^2x\,\epsilon^{\alpha\beta\gamma}\left(A_\alpha^*\partial_\beta A_\gamma-\frac{2i}{3}A_\alpha^*A_\beta^*A_\gamma\right), \qquad (1)$$

where the * product is defined by

$$f(x)*g(x)\equiv[e^{-i(\theta/2)\epsilon^{ij}\partial_i\partial'_j}f(x)g(x')]|_{x=x'}. \qquad (2)$$

Here and below θ will be taken to be positive without loss of generality. One may present the theory in terms of operators on the Hilbert space defined by

$$[\hat{x},\hat{y}]=-i\theta, \qquad (3)$$

where the * product between functions becomes the ordinary product between the operators. For a given function

$$f(x,y)=\int\frac{d^2k}{(2\pi)^2}\tilde{f}(k)e^{i(k_xx+k_yy)}, \qquad (4)$$

the corresponding operator can be found by the Weyl-ordered form of

$$\hat{f}(\hat{x},\hat{y})=\int\frac{d^2k}{(2\pi)^2}\tilde{f}(k)e^{i(k_x\hat{x}+k_y\hat{y})}. \qquad (5)$$

From the mapping, one may easily verify that $\int d^2xf$ is replaced by $2\pi\theta{\rm tr}\hat{f}$ and ∂_if corresponds to $-(i/\theta)\epsilon_{ij}[\hat{x}_j,\hat{f}]$. Let us introduce the creation and annihilation operators by $c^\dagger\equiv(1/\sqrt{2\theta})(x+iy)$ and by $c\equiv(1/\sqrt{2\theta})(x-iy)$, which satisfy $[c,c^\dagger]=1$. In order to represent arbitrary operators we shall use the occupation number basis by $O=\Sigma O_{kl}|k\rangle\langle l|$ with the number operator $c^\dagger c$.

In terms of operator-valued fields, the action can be equivalently written as

$$L_{\rm CS}=-\pi\kappa\theta\epsilon^{\alpha\beta\gamma}{\rm tr}\left(A_\alpha\partial_\beta A_\gamma-\frac{2i}{3}A_\alpha A_\beta A_\gamma\right), \qquad (6)$$

where hats are dropped for simplicity. We introduce the covariant position operator X_i,

$$X_i=x_i-\theta\epsilon_{ij}A_j. \qquad (7)$$

These operators then transform covariantly, i.e., $X'_i=U^\dagger X_iU$ under the gauge transformation

$$A_\mu\rightarrow A'_\mu=iU^\dagger\partial_\mu U+U^\dagger A_\mu U, \qquad (8)$$

with U satisfying $UU^\dagger=U^\dagger U=I$. Written in terms of the covariant position operator, the Lagrangian becomes

$$L_{\rm CS}=-\frac{\pi\kappa}{\theta}{\rm tr}\{-\epsilon_{ij}X_i(\dot{X}_j-i[A_0,X_j])+2\theta A_0\}, \qquad (9)$$

where we dropped total derivative terms. As discussed in Ref. [29], the position X_i describes positions of fluid particles labeled by the the comoving coordinates x_i. The action is invariant under the noncommutative gauge transformation whose continuum version corresponds to the area preserving diffeomorphism. Since x_i is just labels of fluid particles, the theory describing fluid should be invariant under the relabeling as far as the x_i space (labeling) density of fluid particles is preserved. The above Lagrangian describes charged incompressible fluid in the low energies. The noncommutativity parameter θ is related to the the comoving space density ρ_0 of the liquid by

$$\rho_0=\frac{1}{2\pi\theta}. \qquad (10)$$

Thus in some sense, $2\pi\theta$ is the minimal area for a fluid particle occupying in the comoving space x_i. The parameter κ is related to the filling fraction by

$$\nu\equiv\frac{2\pi\rho_0}{\mathcal{B}}=\frac{1}{2\pi\kappa}, \qquad (11)$$

where $\mathcal{B}$ describes the external magnetic field applied to the quantum Hall liquid with the unit electric charge set to unity.

We shall first consider a nonrelativistic matter field minimally coupled to the noncommutative Chern-Simons term. The system is described by

$$L=L_{\rm CS}+2\pi\theta{\rm tr}\left(iD_0\phi\phi^\dagger-\frac{1}{2}D_i\phi(D_i\phi)^\dagger+\frac{1}{2|\kappa|}(\phi\phi^\dagger)^2\right), \tag{12}$$

where we use the fundamental coupling defined by $D_\mu\phi\equiv\partial_\mu\phi-iA_\mu\phi$. This corresponds to a noncommutative version of the Jackiw-Pi model. As in the case of the commutative version of Jackiw-Pi model, this theory allows the BPS bound. To show this, let us first note that the Hamiltonian is given by

$$E=2\pi\theta\,{\rm tr}\left(\frac{1}{2}D_i\phi(D_i\phi)^\dagger-\frac{1}{2|\kappa|}(\phi\phi^\dagger)^2\right). \tag{13}$$

This can be rewritten as

$$E=2\pi\theta{\rm tr}\left(\frac{1}{2}(D_1\pm iD_2)\phi[(D_1\pm iD_2)\phi]^\dagger\pm\frac{1}{2}\epsilon_{ij}D_iJ^j+\frac{1}{2|\kappa|}(\pm|\kappa|B-\phi\phi^\dagger)\phi\phi^\dagger\right), \tag{14}$$

where $J^i=(1/2i)[D_i\phi\phi^\dagger-\phi(D_i\phi)^\dagger]$.

Noting the Gauss law

$$\kappa B=\phi\phi^\dagger, \tag{15}$$

one recognizes that the theory allows the BPS bounds. When $\kappa>0$ the BPS equation becomes

$$(D_1+iD_2)\phi=0 \tag{16}$$

while

$$(D_1-iD_2)\phi=0 \tag{17}$$

for $\kappa<0$. Since the commutation relation (3) breaks the parity symmetry, there is no obvious mapping between these two BPS equations. We find that the former allows a simple localized solution. Hence we shall mainly focus on this case.

Using the covariant position operator, the former BPS equations can be written as

$$K^\dagger\phi-\phi c^\dagger=0,\quad 1-[K,K^\dagger]=\frac{\theta}{\kappa}\phi\phi^\dagger, \tag{18}$$

where we define $K\equiv(X_1-iX_2)/\sqrt{2\theta}$. These BPS equations do not involve A_0. However, once the BPS equations are solved, A_0 can be determined from the equation of motion

$$\kappa\epsilon_{ij}F_{0j}=J_i\,. \tag{19}$$

Together with the BPS equations, one finds

$$A_0=-\frac{\phi\phi^\dagger}{2|\kappa|}, \tag{20}$$

which agrees with the expression of the commutative case except for the specific ordering of the field operators.

III. CLOSELY PACKED MULTIVORTEX SOLUTIONS

The simple m-vortex solutions we found are

$$\phi=\sqrt{\frac{m\kappa}{\theta}}|m-1\rangle\langle 0|,\quad K=P_mcP_m+S_mcS_m^\dagger, \tag{21}$$

where we define the shift operator S_m and the projection operator, respectively, by

$$S_m=\sum_{n=0}^{\infty}|n+m\rangle\langle n|,\quad P_m=\sum_{a=0}^{m-1}|a\rangle\langle a|. \tag{22}$$

The magnetic field of the solution takes the form

$$B=\frac{m}{\theta}|m-1\rangle\langle m-1|. \tag{23}$$

The flux carried by the vortex is $\Phi(\equiv\theta{\rm tr}B)=m$. Hence the solution describes m vortex solutions. The matter density

$$\rho\equiv\phi\phi^\dagger=\frac{m\kappa}{\theta}|m-1\rangle\langle m-1|, \tag{24}$$

is well localized exponentially. This can be seen clearly if one maps the operator to an ordinary function. The density is mapped to

$$\rho=\frac{2m\kappa}{\theta}(-1)^{m-1}L_{m-1}\left(\frac{2r^2}{\theta}\right)e^{-r^2/\theta}, \tag{25}$$

where $r^2=x_ix_i$ and $L_n(x)$ is the nth order Laguerre polynomial. This is quite contrasted to the behaviors of the commutative version of the Jackiw-Pi solitons where the density falls off only by a certain power of the radius. The magnetic field takes quite different form from that of the exact noncommutative solitons of the Abelian Higgs model where $B=(1/\theta)P_m$. Namely, in case of the Abelian Higgs model, the field with total flux m is evenly distributed over from $|0\rangle\langle 0|$ to $|m-1\rangle\langle m-1|$ while it is concentrated on one component in the case of the above Chern-Simons vortex.

To study the properties of these solitons, let us first consider the size of soliton configuration. This can be measured in terms of the covariant position operator. The center position is defined as

$$R^i=\frac{{\rm tr}\,X^i\rho}{{\rm tr}\,\rho}. \tag{26}$$

This definition is gauge invariant whereas ${\rm tr}x^i\rho/{\rm tr}\rho$ does not lead to a gauge invariant quantity. Evaluating the center position of the above solution, one finds that R^i is vanishing for all m. Translations of the vortex solutions give solutions describing the vortices located at the other position. Under a translation, the fields transform as

$$\delta A_i=a^j\partial_jA_i\,,\quad \delta\phi=a^i\partial_i\phi. \tag{27}$$

We add here the gauge transformation in order to make the changes covariant [33]. In particular, we add the transformation by the gauge parameter $\delta\Lambda=-a_iA_i$, whose form coincides with that of the ordinary gauge theory. Hence the translation is described by the transformation

$$\tilde{\delta}A_i = a^j \epsilon_{ji} B, \quad \tilde{\delta}\phi = a^i D_i \phi. \tag{28}$$

The finite form of translation constructed in this manner is in general quite complicated although the corrections occurs only in $m\times m$ and $m\times\infty$ sectors of K and ϕ, respectively. When $m=1$, the expressions of the translated vortex becomes particularly simple. It is

$$\phi = \sqrt{\frac{\kappa}{\theta}}|0\rangle\langle \mathbf{a}|,$$

$$K = S_1 c S_1^\dagger + \sqrt{2\theta}(a_1 - i a_2)|0\rangle\langle 0|, \tag{29}$$

where $|\mathbf{a}\rangle$ denotes a coherent state defined by $|\mathbf{a}\rangle \equiv e^{(i/\theta)\epsilon_{ij}a_i x_j}|0\rangle$. In fact one may generate solutions located at different positions for a general m with a choice of gauge parameter $\delta\Lambda = -(1/\theta)a_i\epsilon_{ij}x_j$, which is gauge equivalent to the above construction. The solution reads

$$\phi = \sqrt{\frac{m\kappa}{\theta}}|m-1\rangle\langle \mathbf{a}|, \quad K = P_m c P_m + S_m c S_m^\dagger + \sqrt{2\theta}(a_1 - i a_2)P_m, \tag{30}$$

with the center position $R_i = a_i$. One may also estimate the size Δ of the solitons by evaluating

$$\Delta^2 \equiv \frac{\mathrm{tr}(X^i - R^i)(X^i - R^i)\rho}{\mathrm{tr}\,\rho}. \tag{31}$$

One then finds that $\Delta^2 = \theta(m-1)$ on the solutions (30). As we will see later on, the size here is minimal in some sense. Namely, the vortex cannot be squeezed further. Thus the minimal distance between vortices are of order $\sqrt{\theta}$. Furthermore, the minimal area taken by m vortices scales roughly as $m\theta$. The behaviors are quite contrasted to the noncommutative vortices in the Abelian Higgs model [8,14] where any number of the noncommutative vortices can be at the same position at least classically. This seems to be closely related to the area preserving nature of the underlying quantum Hall liquid.

The system is invariant under the rotation where the rotation group is $SO(2)$. The parity is broken, which can be seen from the commutation relation (3). Thus angular momentum may be defined by following the Noether construction of the conserved charges. When we use the ordinary function description with the * product, the system is invariant under the rotation

$$\delta A_i = -x_k \epsilon_{kj}\partial_j A_i - \epsilon_{ij}A_j, \quad \delta\phi = -x_k\epsilon_{kj}\partial_j\phi. \tag{32}$$

This can be rewritten, in terms of * product, as

$$\delta A_i = -\frac{1}{2}(x_k^* \epsilon_{kj}\partial_j A_i + \partial_j A_i^* x_k \epsilon_{kj}) - \epsilon_{ij}A_j,$$

$$\delta\phi = -\frac{1}{2}(x_k \epsilon_{kj}^* \partial_j \phi + \partial_j \phi^* x_k \epsilon_{kj}), \tag{33}$$

where we have used the identity $2x_i f(x) = x_i^* f(x) + f(x)^* x_i$ in the description of the ordinary functions. Back to the operator formulation, the rotations read

$$\delta X_i = -\frac{i}{2\theta}[x_k x_k, X_i] - \epsilon_{ij}X_j,$$

$$\delta A_0 = -\frac{i}{2\theta}[x_k x_k, A_0],$$

$$\delta\phi = -\frac{i}{2\theta}[x_k x_k, \phi]. \tag{34}$$

We again add a gauge transformation $\delta\Lambda = -(1/2\theta)(X_k X_k - x_k x_k)$ to make the variation covariant [33]. The resulting transformation of fields are

$$\tilde{\delta}X = -\frac{i}{2\theta}[X_k X_k, X_i] - \epsilon_{ij}X_j,$$

$$\tilde{\delta}A_0 = -\frac{1}{2\theta}D_0(X_k X_k),$$

$$\tilde{\delta}\phi = -\frac{i}{2\theta}(X_k X_k \phi - \phi x_k x_k). \tag{35}$$

Note that the variation $\tilde{\delta}\phi$ is covariant too here. The transformation of X_i can be written in terms of the variation of A_i as $\tilde{\delta}A_i = \frac{1}{2}(X_i B + B X_i)$ that is reduced to the expression for the covariant variation in the commutative gauge theory. The construction of the corresponding Noether charge is straightforward. The resulting expression for the angular momentum is

$$Q_J = 2\pi\theta\,\mathrm{tr}\left(\epsilon_{ij}X_i J^j - \frac{\theta}{2}D_i\phi(D_i\phi)^\dagger\right). \tag{36}$$

In the commutative limit in which the second term on the right side of the above equation is dropped out, this expression reduces to the angular momentum of the commutative theory in Ref. [19]. Now let us evaluate the angular momentum using the solution. One finds

$$Q_J = \pi\kappa m(m-2). \tag{37}$$

The angular momentum scales as the vortex number squared unlike the case of ordinary Jackiw-Pi soliton where it scales linearly in $m-1$. Rather this agrees to the angular momentum found in the ordinary relativistic model [34], which reads

$$Q_J' = \pi\kappa\Phi(2N - \Phi), \tag{38}$$

with N denoting the vorticity. Since $N = m-1$ and $\Phi = m$ for our solutions, one finds the agreement. In the next section, we shall show that the orbital part alone is just $\pi\kappa m(m-1)$ by studying the separation of vortices.

As will be shown later, there is no further solution of $m=1$ within rotationally symmetric ansatz. Hence the $m=1$ case above is a unique solution that is rotationally symmetric. Let us consider possible moduli of the $m=1$ solution. In the commutative case, there are three moduli parameters for the one vortex solution. Two are locations of a vortex and the remaining is a scale parameter. All of these are related to symmetry transformations. Namely, the position is generated by the translation while an arbitrary size can be generated by a scale transformation that is also a symmetry of the model. In the noncommutative case, we have already found two moduli parameters representing the positions of the vortex. The question is what happens to the scale parameter. Because of the noncommutative scale on which the theory is explicitly dependent, the scale symmetry is no longer a symmetry of the model. But one may still generate solutions for a given solution by the scale transformation

$$x_i \to b x_i, \quad t \to b^2 t \tag{39}$$

together with $\theta \to b^2\theta$, where the factor b^2 can be understood from the fact that the noncommutativity scale has a dimension of length squared. Unfortunately, the above m-vortex solutions are invariant under this scale transformation and no new solutions are generated.[2] Hence it seems plausible that the $m=1$ solution has only two moduli parameters as the moduli space of one soliton is in general generated by a symmetry transformation.

IV. SEPARATION OF TWO VORTICES

In order to study the separation of two vortices, we first note that the translations of the m-coincident vortices in the last section generate a nontrivial modification of fields only in $m\times m$ and $m\times\infty$ sectors of K and ϕ, respectively. Since the separation can be understood as a translation of each vortex, we take an ansatz where

$$\phi = \sum_{a=0}^{m-1} |a\rangle\langle\psi_a|, \quad K = \sum_{a=0}^{m-1}\sum_{b=0}^{m-1} V_{ab}|a\rangle\langle b| + S_m c S_m^\dagger, \tag{40}$$

where $|\psi_a\rangle$ refers to a generic state in the Hilbert space defined by $[x,y]=-i\theta$. The BPS equation reduces to

$$V^\dagger\phi - \phi c^\dagger = 0, \quad P_m - [V,V^\dagger] = \frac{\theta}{\kappa}\phi\phi^\dagger. \tag{41}$$

Now let us consider $m=2$ case. We choose a gauge where B is diagonal. The the most general form of $V^\dagger$ takes

$$V^\dagger = q_0 P_2 + q_1|0\rangle\langle 1| + q_2|1\rangle\langle 0|, \tag{42}$$

where q_0, q_1, and q_2 are complex numbers. Note that under a transformation

$$\phi \to \phi e^{c^\dagger q_0^* - c q_0}, \quad V^\dagger \to V^\dagger - q_0 P_2, \tag{43}$$

the BPS equation is invariant, which corresponds to a translation. Hence one may set $q_0=0$ without loss of generality. Solving the first BPS equation is now straightforward. For $\phi_{00}=\lambda_1$ and $\phi_{10}=\lambda_2$, the general solution reads

$$\langle\psi_1| = \lambda_1\langle \mathrm{ch}| + \lambda_2 q_1\langle \mathrm{sh}|, \quad \langle\psi_2| = \lambda_1 q_2\langle \mathrm{sh}| + \lambda_2\langle \mathrm{ch}|, \tag{44}$$

where we define

$$\langle \mathrm{sh}| = \langle 1| + \frac{q_1q_2}{\sqrt{3!}}\langle 3| + \frac{(q_1q_2)^2}{\sqrt{5!}}\langle 5| + \cdots,$$

$$\langle \mathrm{ch}| = \langle 0| + \frac{q_1q_2}{\sqrt{2!}}\langle 2| + \frac{(q_1q_2)^2}{\sqrt{4!}}\langle 4| + \cdots. \tag{45}$$

Since we take the gauge where the magnetic field is diagonal, $\langle\psi_1|\psi_2\rangle$ ought to vanish, which can be seen from the second BPS equation. This implies either λ_1 or λ_2 is zero. We take $\lambda_1=0$ and $\lambda_2=\lambda$ using the residual gauge symmetry that interchanges $|0\rangle$ and $|1\rangle$. One may further choose λ to be real and positive. Inserting the expressions to the second BPS equation, we obtain a set of algebraic equations

$$B_{00} = \frac{1}{\theta}(1+|q_1|^2-|q_2|^2) = \frac{\lambda^2}{\kappa}\left|\frac{q_1}{q_2}\right|\sinh|q_1q_2|,$$

$$B_{11} = \frac{1}{\theta}(1-|q_1|^2+|q_2|^2) = \frac{\lambda^2}{\kappa}\cosh|q_1q_2|. \tag{46}$$

Dividing the first equation by the second equation, we get

$$\frac{1+|q_1|^2-|q_2|^2}{1-|q_1|^2+|q_2|^2} = \left|\frac{q_1}{q_2}\right|\tanh|q_1q_2|, \tag{47}$$

from which one may find $|q_2|$ for an arbitrary $|q_1|\in[0,\infty)$. Once we get $|q_2|$ for a given $|q_1|$, λ can be determined from any of the above equation. When $|q_1|=0$, one finds $|q_2|=1$, which corresponds to the closely packed two vortex solution described in the previous section. From the equation it is obvious that the maximum of $||q_1|^2-|q_2|^2|$ equals 1. Let us now show that the minimum of $|q_1|^2+|q_2|^2$ is also 1. To show this, let us assume that $|q_1|^2+|q_2|^2<1$. Then from the left-hand side of the above equation, one finds that

$$\frac{1+|q_1|^2-|q_2|^2}{1-|q_1|^2+|q_2|^2} > \frac{2|q_1|^2}{1-|q_1|^2+|q_2|^2} > \frac{|q_1|^2}{1-|q_1|^2} > |q_1|^2. \tag{48}$$

Noting $\tanh|q_1q_2|/|q_1q_2|<1$, we find that the right-hand side is smaller than $|q_1|^2$. Hence we get a contradiction. Similarly, one may easily show that $|q_1|<|q_2|$ on the solution for all

[2]For example, $|0\rangle\langle 0|$ is mapped to $2e^{-r^2/\theta}$ and it is clear that it is invariant under the scale transformation together with the change of θ.

$|q_1|$. Furthermore, when $|q_1|$ become large, the difference between $|q_2|$ and $|q_1|$ becomes exponentially small as $\sim e^{-2|q_1|^2}/2|q_1|$.

The overall position R_i of this solution is zero. The size of the configuration is

$$\Delta^2=\theta(|q_1|^2+|q_2|^2)\geqslant\theta. \tag{49}$$

The last inequality follows from the fact that the minimum of $(|q_1|^2+|q_2|^2)$ is one as shown above. We further compute the higher moments

$$\frac{\mathrm{tr}[(X^i-R^i)(X^i-R^i)]^n\rho}{\mathrm{tr}\,\rho}=[\theta(|q_1|^2+|q_2|^2)]^n. \tag{50}$$

In the commutative theory, this kind of behavior for the higher moments can be found only when two pointlike masses of an equal mass are separated by a distance

$$d=2\sqrt{\theta(|q_1|^2+|q_2|^2)}. \tag{51}$$

Hence we conclude that the configuration describes two separated vortices at a distance d. The angular momentum of the configuration can be found as

$$Q_J=2\pi\kappa[(|q_1|^2-|q_2|^2)^2-1]. \tag{52}$$

The maximum of the angular momentum occurs at the closely packed solutions in the previous section. The angular momentum decreases as the separation gets larger and becomes twice of the angular momentum of one vortex of $\Phi=1$ in the large separation limit. Therefore one may consistently regard each vortex carrying an intrinsic angular momentum $-\pi\kappa$. The orbital part of the angular momentum in the two vortex case is

$$Q_{\mathrm{orb}}=Q_J+m\pi\kappa=2\pi\kappa(|q_1|^2-|q_2|^2)^2. \tag{53}$$

Thus as claimed before, the orbital part of angular momentum becomes $\pi\kappa m(m-1)$ for the closely packed m vortices.

Finally let us count moduli parameters of these separated vortices. There are two real parameters describing the center position specified by a complex number q_0. $|q_2|$ is fixed by $|q_1|$ and the real parameter λ is completely fixed in terms of $|q_1|$. Among the argument of q_1 and q_2, the relative part $(\varphi_1-\varphi_2)$ can be gauged away and only the overall part is gauge invariant. In fact, under the rotation defined in Eq. (35), q_1 and q_2 change as $q_1'=q_1e^{i\varphi}$ $q_2'=q_2e^{i\varphi}$. Thus we conclude that the overall argument describes the angular coordinates in the relative space of the two vortices. In total we have four real moduli parameters which specify the planar locations of two vortices.

V. GENERAL ROTATIONALLY SYMMETRIC VORTICES

The rotationally symmetric configuration of the noncommutative Jackiw-Pi model was studied numerically in Ref. [26]. Nonetheless we repeat here the analysis to show certain analytic properties. We first consider the BPS equation with $\kappa>0$. We take a generic rotationally symmetric ansatz

$$\phi=\sum_{n=0}^{\infty}\phi_n|n+m-1\rangle\langle n|,\quad K^\dagger=\sum_{n=0}^{\infty}k_n^*|n+1\rangle\langle n|, \tag{54}$$

for some positive integer m. One may easily verify that the other choice $\phi=\Sigma_{n=0}^{\infty}\phi_n|n\rangle\langle n+m|$ does not lead to any solutions. The BPS equations then imply that

$$\begin{aligned}&\phi_{n+1}\sqrt{n+1}=\phi_n k^*_{n+m-1},\\ &B_{n+m-1}=\frac{1}{\theta}(1-|k_{n+m-1}|^2+|k_{n+m-2}|^2)=\frac{1}{\kappa}|\phi_n|^2,\end{aligned} \tag{55}$$

for $n\geqslant0$ with

$$|k_a|=a+1,\ B_a=0\ \text{ for }\ a<m-1. \tag{56}$$

Defining $|k_{n+m-1}|=n+v_n\quad(\geqslant0)$, one gets a recurrence relation

$$v_{n+1}-v_n=\frac{n+v_n}{n+1}(v_n-v_{n-1}), \tag{57}$$

for $n\geqslant1$. The initial conditions are identified from Eq. (55) as $0\leqslant v_0\leqslant m$ and $v_1-v_0=-v_0(m-v_0)$. Since $v_1-v_0\leqslant0$, v_n-v_{n+1} is all nonnegative definite. Hence the series v_n is either constant or monotonically decreasing. The total flux here is given by $\Phi=m-v_\infty$.

There are two special points where v_n is constant. When $v_0=m$, $v_n=m$ and $\phi_n=0$ for all n. The flux is zero and this corresponds to the trivial vacuum solution. When $v_0=0$, $v_n=0$ and $|\phi_n|^2$ is nonvanishing only for $n=m-1$ with a value $m\kappa/\theta$. This is the closely packed solution we discussed in the previous section. Except these two cases, v_∞ must be negative definite. To show this, let us assume $v_\infty\geqslant0$. Since v_n is monotonically decreasing, v_n-v_{n+1} is then positive definite. From the recurrence relation (57), we have

$$v_n-v_{n+1}>\frac{n}{n+1}(v_{n-1}-v_n)>\cdots>\frac{1}{n+1}(v_0-v_1)>0 \tag{58}$$

for $n\geqslant1$. Summing this for $l=1$ to n, one finds

$$v_1-v_{n+1}>(v_0-v_1)\sum_{l=1}^{n}\frac{1}{l+1}. \tag{59}$$

For large n, the right-hand side diverges as $\ln n$ and we get a contradiction. Hence we get a bound for the total flux as

$$\Phi=m-v_\infty>m \tag{60}$$

when $v_0\neq0,m$. This proves the claim that the closely packed $\Phi=1$ solution is unique within the rotationally symmetric ansatz. The series for $0<v_0<m$ case in general converges. Whether the flux is quantized or not for the generic initial data seems not clear though in Ref. [26] it is claimed that they found that it is not quantized numerically. In case of the commutative Jackiw-Pi model, such solutions carrying noninteger flux are excluded by the analyticity requirement al-

though the solutions satisfy locally the equations of motion. We do not know how to implement such analyticity in the noncommutative case. In this respect, the quantization of the flux numbers are not quite clear. We leave this issue for the future study.

Now let us turn to the case of $\kappa<0$. In this case an appropriate ansatz will be

$$\phi=\sum_{n=0}^{\infty}\phi_n|n\rangle\langle n+m-1|,\quad K^\dagger=\sum_{n=0}^{\infty}k_n^*|n+1\rangle\langle n|. \tag{61}$$

The other choice $\phi=\Sigma_{n=0}^{\infty}\phi_n|n+m\rangle\langle n|$ again does not lead to any solutions. Inserting the ansatz to the BPS equations for $\kappa<0$, one gets the recurrence relation

$$\phi_{n+1}k_n=\phi_n\sqrt{n+m},$$

$$B_n=\frac{1}{\theta}(1-|k_n|^2+|k_{n-1}|^2)=-\frac{1}{|\kappa|}|\phi_n|^2 \tag{62}$$

for $n\geqslant 0$ with $k_{-1}=0$. Eliminating ϕ_n again, we get

$$v_{n+1}-v_n=\frac{n+m}{n+v_n}(v_n-v_{n-1}), \tag{63}$$

where we define again $|k_n|^2=n+v_n\geqslant 0$. The total flux is then given by $\Phi=1-v_\infty$. The initial conditions read $v_0\geqslant 1$ and $v_1-v_0=m(v_0-1)/v_0\geqslant 0$. When $v_0=1$, we get the trivial vacuum solution again, so we shall consider only the case of $v_0>1$. v_n is monotonically increasing and one may easily show that $v_\infty>m+1$ by a similar way adopted for $\kappa>0$. Hence we conclude that any nontrivial vortex solutions carry the flux $\Phi<-1$. In summary we have proved that, within the rotationally symmetric ansatz, the $\Phi=1$ solution is unique and any other solutions should necessarily carry the flux with $|\Phi|>1$.

VI. NONTOPOLOGICAL VORTICES IN THE RELATIVISTIC CHERN-SIMONS MODEL

In this section we shall consider the relativistic Chern-Simons model with fundamentally coupled matter described by

$$L=L_{\mathrm{CS}}+2\pi\theta\,\mathrm{tr}[D_0\phi(D_0\phi)^\dagger-D_i\phi(D_i\phi)^\dagger-V(\phi\phi^\dagger)], \tag{64}$$

where the potential is

$$V(\xi)=\frac{1}{\kappa^2}\xi(v^2-\xi)^2. \tag{65}$$

We shall derive here the BPS equation similarly to the case of the nonrelativistic model. The derivation was done in Ref. [26], but we find it incomplete because they assume certain relations that are unnecessary. To begin with, let us first note that the Hamiltonian is given by

$$E=2\pi\theta\,\mathrm{tr}[D_0\phi(D_0\phi)^\dagger+D_i\phi(D_i\phi)^\dagger+V(\phi\phi^\dagger)]. \tag{66}$$

We simply note that the above Hamiltonian can be rewritten identically as

$$E=2\pi\theta\mathrm{tr}\left[\left|D_0\phi\pm\frac{i}{\kappa}(v^2-\phi\phi^\dagger)\phi\right|^2+|(D_1\pm iD_2)\phi|^2\right.$$
$$\left.\pm(\phi\phi^\dagger-v^2)\left(B-\frac{j_0}{\kappa}\right)\mp\frac{\epsilon_{mn}}{2}D_mj_n\pm v^2B\right], \tag{67}$$

where we define the current j_μ as

$$j_\mu=i[D_\mu\phi\phi^\dagger-\phi(D_\mu\phi)^\dagger], \tag{68}$$

and $|O|^2\equiv OO^\dagger$ for any operator O. Now using the Gauss law constraint

$$\kappa B=j_0, \tag{69}$$

and ignoring the total derivative term, we find

$$E=2\pi\theta\,\mathrm{tr}\left(\left|D_0\phi\pm\frac{i}{\kappa}(v^2-\phi\phi^\dagger)\phi\right|^2\right.$$
$$\left.+|(D_1\pm iD_2)\phi|^2\pm v^2B\right)\geqslant v^2|\Phi|. \tag{70}$$

The saturation of the bound occurs if the BPS equations

$$(D_1\pm iD_2)\phi=0,$$

$$D_0\phi\pm\frac{i}{\kappa}(v^2-\phi\phi^\dagger)\phi=0 \tag{71}$$

hold. Using again the Gauss law, the second equation can be rewritten as

$$B=\pm\frac{2}{\kappa^2}\phi\phi^\dagger(v^2-\phi\phi^\dagger). \tag{72}$$

We shall not explore the full details of the soliton solutions present in this model. Instead, we shall focus on the nontopological soliton solutions. For the closely packed solutions, one has two types of solutions for the same magnetic field: They are

$$\phi=\lambda_\pm|m-1\rangle\langle 0|,\quad K=P_mcP_m+S_mcS_m^\dagger, \tag{73}$$

where $\lambda_\pm$ is given by

$$\lambda_\pm^2=v^2\left[1\pm\left(1-\frac{2m\kappa}{\theta v^4}\right)^{1/2}\right]. \tag{74}$$

The solution of this type exists only for $\theta\geqslant 2m\kappa/v^4$. Hence it is clear that the solution do not exist in the commutative limit where θ goes to zero. The flux is again $\Phi=m$ and the energy is $2\pi m v^2$ that of course saturates the BPS bound. The charge carried by the solution is $Q=2\pi\theta\mathrm{tr}j_0=2\pi\kappa m$. The position is defined as

$$R_i \equiv \frac{\mathrm{tr}\, X_i j_0}{\mathrm{tr}\, j_0}, \tag{75}$$

and the center position vanishes on the solution. The size defined with respect to the charge density is $\Delta^2 = \theta(m-1)$.

The position can be defined with respect to the Hamiltonian density $\mathcal{H}$:[3]

$$R_H^i \equiv \frac{\mathrm{tr}\, X^i \mathcal{H}}{\mathrm{tr}\, \mathcal{H}}, \tag{76}$$

which vanishes on the solution. One may also consider the matter distribution size as

$$\Delta_H^2 \equiv \frac{\mathrm{tr}(X^i - R_H^i)(X^i - R_H^i)\mathcal{H}}{\mathrm{tr}\, \mathcal{H}}. \tag{77}$$

For the vortex configuration, this size of the energy distribution is found to be

$$\Delta_H^2 = (m-1)\theta\left(1 + \frac{(m-2)\lambda_\pm^2}{mv^2}\right). \tag{78}$$

The value is again zero for $m=1$ and there is no difference between the two branches for $m=2$. For $m>2$, the $(-)$ branch solution has more closely packed; namely, the minimal intervortex distance is smaller in the $(-)$ branch.

The angular momentum can be found in a similar way to the nonrelativistic case. Under the rotation in Eq. (35), the Noether procedure leads to the angular momentum

$$Q_J = 2\pi\theta\, \mathrm{tr}\left(\epsilon_{ij} X_i T^{0j} - \frac{i\theta}{2}[D_i D_0 \phi (D_i \phi)^\dagger - D_i \phi (D_i D_0 \phi)^\dagger]\right), \tag{79}$$

where T^{0i} is the momentum density

$$T^{0i} = -\frac{1}{2}[D_i \phi (D_0 \phi)^\dagger + D_0 \phi (D_i \phi)^\dagger]. \tag{80}$$

The expression again reduces to that of the ordinary field theory in the commutative limit. The angular momentum evaluated on the above solution is $\pi\kappa m(m-2)$. Hence the solutions describe the closely packed m vortices and their properties are quite similar to those of nonrelativistic counterpart. One thing that is distinct from the nonrelativistic closely packed solution lies in the fact that there are two types of solutions for the same given magnetic field configuration. In the $(-)$ branch solution, nonvanishing components of $(\phi\phi^\dagger)_{nn}$ are centered around the symmetric vacuum while the components of $(+)$ branch solution are centered around the broken vacuum. As θ grows, the values of $\lambda_\pm$ approach the vacuum values. When $\theta v^4 = 2m\kappa$, the two branches coincide and become just one solution.

The separations of two closely packed solutions can be separated again. The solution takes a form

$$\phi = \lambda(q_1|0\rangle\langle \mathrm{sh}| + |1\rangle\langle \mathrm{ch}|),$$
$$K^\dagger = q_1|0\rangle\langle 1| + q_2|1\rangle\langle 0| + S_m c^\dagger S_m^\dagger. \tag{81}$$

The diagonal part of the 2×2 sector of $K^\dagger$ again describes the center position and we set it to zero using the translational symmetry. The parameters satisfy the algebraic equations

$$B_{00} = \frac{1}{\theta}(1 + |q_1|^2 - |q_2|^2) = \frac{2\lambda^2}{\kappa^2}\left|\frac{q_1}{q_2}\right| \sinh|q_1 q_2| \left(v^2 - \lambda^2 \left|\frac{q_1}{q_2}\right| \sinh|q_1 q_2|\right),$$

$$B_{11} = \frac{1}{\theta}(1 - |q_1|^2 + |q_2|^2) = \frac{2\lambda^2}{\kappa^2} \cosh|q_1 q_2| (v^2 - \lambda^2 \cosh|q_1 q_2|). \tag{82}$$

The number of parameters involved are again four real parameters. The center position has two real parameters and $|q_1|$ and the overall angle of q_1 and q_2 are the remaining moduli. The relative angle can be gauged away and $|q_2|$ and λ are determined interms of $|q_1|$ by the above set of the equations.

VII. CONCLUSION

In this paper we have constructed the solutions of the noncommutative Chern-Simons solitons and investigated their properties by computing the position, the size and the angular momentum. The nature of the noncommutative vortices differ from those in the commutative version.

In the case of the closely packed multi-vortex solutions, we have considered their moduli separations only for the case of two vortices. For the general closely packed vortices with $m>2$, the ansatz taken in this paper leads to a finite set of closed algebraic equations. Though there might arise a little complication, it is worth studying the detailed structure of their moduli dependence. In particular how the minimal size m-vortex configurations are achieved in a moduli dependent manner would be quite interesting. On the nonrelativistic model, one problem unresolved is whether there is a quantization of flux generically. In Ref. [26], it is suggested that the flux quantization of the vortex solutions does not occur if one just require the convergence of the difference equation. In case of the commutative Jackiw-Pi model, only the regularity requirement of the solution at the origin leads to a quantization. We do not know how to understand such conditions in the noncommutative case. Further study is required on the issue. The other issue we omitted in this paper is on the low energy moduli dynamics. In the commutative

[3]We define here the Hamiltonian density operator as the covariant quantity inside the trace of Eq. (66).

Jackiw-Pi model, it is shown that the solitons behave as dual object of the elementary particle excitations [35]. How the dynamics occurs in case of the noncommutative vortices need to be clarified. Also in this respect, the quantum nature of elementary excitation would be of interest [36].

In case of the relativistic model, we have not explored the solitons in the topological sector. The BPS equations suggest that, the solitons in this sector might have some common properties with the Abelian Higgs model since they involve a broken vacuum. There are actually models where the BPS equations are precisely the same as that of the Abelian Higgs model [5,8,14]. They are the nonrelativistic models with repulsive interactions with the background charge density [37] or the external magnetic field [38]. Since some of the properties of solutions are known in this case, the study of the low energy dynamics would be quite interesting.

Note added. The problem in this paper was originally suggested by one of the authors [8]. While we were investigating the subject, a related paper appeared [26]. We found some overlaps in Sec. V.

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